① 電磁気演習 (大川祐司)

問題1

(1) $\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$

$\vec{r}' = \vec{r} - \vec{a} = (x, y, z)$ $\quad r' = |\vec{r}'| = \sqrt{x^2+y^2+z^2}$ とする

$\vec{E}(\vec{r}) = \left(-\frac{\partial}{\partial x}\phi(\vec{r}'), -\frac{\partial}{\partial y}\phi(\vec{r}'), -\frac{\partial}{\partial z}\phi(\vec{r}')\right)$

$= \left(-\frac{dr'}{dx}\frac{\partial}{\partial r'}\phi(\vec{r}'), -\frac{dr'}{dy}\frac{\partial}{\partial r'}\phi(\vec{r}'), -\frac{dr'}{dz}\frac{\partial}{\partial r'}\phi(\vec{r}')\right)$

$\frac{\partial r'}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r'}$

同様に $\frac{\partial r'}{\partial y} = \frac{y}{r'}$, $\frac{\partial r'}{\partial z} = \frac{z}{r'}$

$\frac{\partial}{\partial r'}\phi(\vec{r}') = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{|\vec{r}'|^2}$

$\therefore \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(-\frac{qx}{|\vec{r}'|^3}, -\frac{qy}{|\vec{r}'|^3}, -\frac{qz}{|\vec{r}'|^3}\right)$

$= \left(-\frac{1}{4\pi\varepsilon_0}\cdot\frac{qx}{|\vec{r}-\vec{a}|^3}, -\frac{1}{4\pi\varepsilon_0}\cdot\frac{qy}{|\vec{r}-\vec{a}|^3}, -\frac{1}{4\pi\varepsilon_0}\cdot\frac{qz}{|\vec{r}-\vec{a}|^3}\right)$

(2) $E(\vec{r}) = (E_x, E_y, E_z)$ とおくと

$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z$

$\frac{\partial}{\partial x}E_x = \frac{\partial}{\partial x}\left(-\frac{1}{4\pi\varepsilon_0}\cdot\frac{qx}{(r')^3}\right) = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r'^3} + \left(\frac{3}{4\pi\varepsilon_0}\frac{qx}{(r')^4}\right)\frac{dr'}{dx}$

$= -\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r'^3} - 3\frac{qx^2}{r'^5}\right)$ — ①

同様に

$\frac{\partial}{\partial y}E_y = -\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r'^3} - 3\frac{qy^2}{r'^5}\right)$ — ②

$\frac{\partial}{\partial z}E_z = -\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r'^3} - 3\frac{qz^2}{r'^5}\right)$ — ③

①, ②, ③ より

$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z$

$= -\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r'^3}\right)\times 3 + \frac{1}{4\pi\varepsilon_0}\times 3\times\left(\frac{qx^2+qy^2+qz^2}{r'^5}\right)$

$= 0$

($\because r'^2 = x^2+y^2+z^2$ より)

$\vec{\nabla}\cdot\vec{E}(\vec{r}) = 0.$

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$$\vec{\nabla}\times\vec{E}(\vec{r}) = \left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z},\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x},\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z} = \frac{\partial}{\partial y}\left(-\frac{1}{4\pi\varepsilon_0}\cdot\frac{qz}{r'^3}\right)-\frac{\partial}{\partial z}\left(-\frac{1}{4\pi\varepsilon_0}\cdot\frac{qy}{r'^3}\right)$$

$$= \frac{dr'}{dy}\left(-\frac{qz}{4\pi\varepsilon_0}\frac{\partial}{\partial r'}\left(\frac{1}{r'^3}\right)\right)-\frac{dr'}{dz}\left\{-\frac{qy}{4\pi\varepsilon_0}\frac{\partial}{\partial r'}\left(\frac{1}{r'^3}\right)\right\}$$

$$= -\frac{q}{4\pi\varepsilon_0}\left(\frac{-3yz}{r'^5}+\frac{3yz}{r'^5}\right) = 0$$

同様に $\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x} = -\frac{q}{4\pi\varepsilon_0}\left(\frac{-3xz}{r'^5}+\frac{3xz}{r'^5}\right) = 0$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y} = -\frac{q}{4\pi\varepsilon_0}\left(\frac{-3xy}{r'^5}+\frac{3xy}{r'^5}\right) = 0$$

$\therefore\ \vec{\nabla}\times\vec{E}(\vec{r}) = (0,0,0)$

問題2. $\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{|\vec{r}|^3}$

$\vec{p} = (0,0,p)$, $\vec{r} = (x,y,z)$ とおく

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{pz}{|\vec{r}|^3}$$

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

$$= \frac{-p}{4\pi\varepsilon_0}\left(\frac{3zx}{|\vec{r}|^5},\ \frac{3zy}{|\vec{r}|^5},\ \frac{3z^2-r^2}{|\vec{r}|^5}\right)$$

$z^2 - r^2$

問題3

(1) $\vec{\nabla}\phi(\vec{r})$ はベクトル場である

ストークスの定理より、空間内に
任意の閉曲面 S が存在すると任意の
空間内のベクトル場 $\vec{E}(\vec{r})$ に対して

$$\int_S \vec{\nabla}\times\vec{E}(\vec{r})\cdot d\vec{S} = \int_{\partial S} \vec{E}(\vec{r})\cdot d\vec{r} = 0$$

$$\therefore \vec{\nabla}\times\vec{E}(\vec{r}) = 0$$

$$\vec{\nabla}\times\vec{\nabla}\phi(\vec{r}) = 0$$

(2) いまスカラー場 $f(\vec{r})$, $g(\vec{r})$ に対して

$$\vec{\nabla}\cdot(f(\vec{r})\vec{\nabla}g(\vec{r}))$$
$$= \vec{\nabla}f(\vec{r})\cdot\vec{\nabla}g(\vec{r}) + f(\vec{r})\nabla^2 g(\vec{r})$$ が成り立つ

$$\therefore f(\vec{r})\nabla^2 g(\vec{r}) = \vec{\nabla}\cdot(f(\vec{r})\vec{\nabla}g(\vec{r})) - \vec{\nabla}f(\vec{r})\cdot\vec{\nabla}g(\vec{r})$$

$$\therefore \int_V \phi\nabla^2\phi\, dV = \int_V \vec{\nabla}\cdot(\phi\vec{\nabla}\phi)\, dV - \int_V |\vec{\nabla}\phi|^2 dV$$

ガウスの定理より 任意の閉空間 V において

$$\int_V \vec{\nabla}\cdot(\phi\vec{\nabla}\phi)\, dV = \int_{\partial V} \phi\vec{\nabla}\phi\cdot d\vec{S}$$

が成り立つので

$$\int_V \phi\nabla^2\phi\, dV = \int_{\partial V}\phi\vec{\nabla}\phi\cdot d\vec{S} - \int_S |\vec{\nabla}\phi|^2 dV$$

問題4

(1) $\vec{A}(\vec{r}) = \left(A_x(\vec{r}),\ A_y(\vec{r}),\ A_z(\vec{r})\right)$ とおく

$$\vec{\nabla}\times\vec{A}(\vec{r}) = \left(\frac{\partial A_z(\vec{r})}{\partial y} - \frac{\partial A_y(\vec{r})}{\partial z},\ \frac{\partial A_x(\vec{r})}{\partial z} - \frac{\partial A_z(\vec{r})}{\partial x},\ \frac{\partial A_y(\vec{r})}{\partial x} - \frac{\partial A_x(\vec{r})}{\partial y}\right)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times A(\vec{r})) = \left(\frac{\partial^2 A_z(\vec{r})}{\partial x\partial y} - \frac{\partial^2 A_y(\vec{r})}{\partial x\partial z} + \frac{\partial^2 A_x(\vec{r})}{\partial y\partial z} - \frac{\partial^2 A_z(x)}{\partial y\partial x} + \frac{\partial^2 A_y(\vec{r})}{\partial z\partial x} - \frac{\partial^2 A_x(\vec{r})}{\partial z\partial y}\right)$$

$$= 0$$

(2)
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{r})) = \vec{\nabla}\times\begin{pmatrix} \frac{\partial A_z(\vec{r})}{\partial y} - \frac{\partial A_y(\vec{r})}{\partial z} \\ \frac{\partial A_x(\vec{r})}{\partial z} - \frac{\partial A_z(\vec{r})}{\partial x} \\ \frac{\partial A_y(\vec{r})}{\partial x} - \frac{\partial A_x(\vec{r})}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial y}\left(\frac{\partial A_y(\vec{r})}{\partial x} - \frac{\partial A_x(\vec{r})}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x(\vec{r})}{\partial z} - \frac{\partial A_z(\vec{r})}{\partial x}\right) \\ \frac{\partial}{\partial z}\left(\frac{\partial A_z(\vec{r})}{\partial y} - \frac{\partial A_y(\vec{r})}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_y(\vec{r})}{\partial x} - \frac{\partial A_x(\vec{r})}{\partial y}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial A_x(\vec{r})}{\partial z} - \frac{\partial A_z(\vec{r})}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z(\vec{r})}{\partial y} - \frac{\partial A_y(\vec{r})}{\partial z}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial^2 A_y(\vec{r})}{\partial y\partial x} + \frac{\partial^2 A_z(\vec{r})}{\partial z\partial x} + \frac{\partial^2 A_x(\vec{r})}{\partial x^2} - \nabla^2 A_x(\vec{r}) \\ \frac{\partial^2 A_z(\vec{r})}{\partial z\partial y} + \frac{\partial^2 A_x(\vec{r})}{\partial x\partial y} + \frac{\partial^2 A_y(\vec{r})}{\partial y^2} - \nabla^2 A_y(\vec{r}) \\ \frac{\partial^2 A_x(\vec{r})}{\partial x\partial z} + \frac{\partial^2 A_y(\vec{r})}{\partial y\partial z} + \frac{\partial^2 A_z(\vec{r})}{\partial z^2} - \nabla^2 A_z(\vec{r}) \end{pmatrix}$$

$$= -\nabla^2\vec{A}(\vec{r}) + \vec{\nabla}\begin{pmatrix} \vec{\nabla}\cdot A_x(\vec{r}) \\ \vec{\nabla}\cdot A_y(\vec{r}) \\ \vec{\nabla}\cdot A_z(\vec{r}) \end{pmatrix}$$

$$= -\nabla^2\vec{A}(\vec{r}) + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}(\vec{r}))$$

問題5 $\vec{r} = (x, y, z)$ とおく.

$$\nabla^2\phi(x) = \frac{\partial^2}{\partial x^2}\phi(r) + \frac{\partial^2}{\partial y^2}\phi(r) + \frac{\partial^2}{\partial z^2}\phi(r)$$

$$= \frac{dr}{dx}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}\phi(r)\right) + \frac{dr}{dy}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial y}\phi(r)\right) + \frac{dr}{dz}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial z}\phi(r)\right)$$

$$= \frac{dr}{dx}\frac{\partial}{\partial r}\left(\frac{dr}{dx}\frac{\partial}{\partial r}\phi(r)\right) + \frac{dr}{dy}\frac{\partial}{\partial r}\left(\frac{dr}{dy}\frac{\partial}{\partial r}\phi(r)\right) + \frac{dr}{dz}\frac{\partial}{\partial r}\left(\frac{dr}{dz}\frac{\partial}{\partial r}\phi(r)\right)$$

$r = \sqrt{x^2+y^2+z^2}$ $\quad \frac{dr}{dx} = \frac{x}{r}, \quad \frac{dr}{dy} = \frac{y}{r}, \quad \frac{dr}{dz} = \frac{z}{r}$

$$\nabla^2\phi(x) = \frac{x}{r}\frac{\partial}{\partial r}\left(\frac{x}{r}\frac{d}{dr}\phi(r)\right) + \frac{y}{r}\frac{\partial}{\partial r}\left(\frac{y}{r}\frac{d}{dr}\phi(r)\right) + \frac{z}{r}\frac{\partial}{\partial r}\left(\frac{z}{r}\frac{d}{dr}\phi(r)\right)$$

$$= \frac{x}{r}\left(\frac{x}{r}\frac{d^2}{dr^2}\phi(r) + \frac{dx}{dr}\frac{1}{r}\frac{d}{dr}\phi(r) + \left(-\frac{x}{r^2}\right)\frac{d}{dr}\phi(r)\right)$$

$$+ \frac{y}{r}\left(\frac{y}{r}\frac{d^2}{dr^2}\phi(r) + \frac{dy}{dr}\frac{1}{r}\frac{d}{dr}\phi(r) + \left(-\frac{y}{r^2}\right)\frac{d}{dr}\phi(r)\right)$$

$$+ \frac{z}{r}\left(\frac{z}{r}\frac{d^2}{dr^2}\phi(r) + \frac{dz}{dr}\frac{1}{r}\frac{d}{dr}\phi(r) + \left(-\frac{z}{r^2}\right)\frac{d}{dr}\phi(r)\right)$$

$$= \left(\frac{x^2+y^2+z^2}{r^2}\right)\frac{d^2}{dr^2}\phi(r) + \frac{1+1+1}{r}\frac{d}{dr}\phi(r) - \frac{x^2+y^2+z^2}{r^3}\frac{d}{dr}\phi(r)$$

$$= \frac{d^2}{dr^2}\phi(r) + \frac{2}{r}\frac{d}{dr}\phi(r)$$

問題6

(1) $$\vec{E}(\vec{r}) = \int_0^{2\pi}\frac{(\vec{r}-\vec{r}')}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|^3}\lambda\, l\, d\theta$$

$(\,|\vec{r}'| = l \quad \vec{r}' = (l\cos\theta, l\sin\theta, 0)\,)$

$\vec{r} = (0, 0, z)$ のとき

$$\vec{E}(\vec{r}) = \left(\int_0^{2\pi}\frac{-l\cos\theta\,\lambda l\,d\theta}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|^3},\ \int_0^{2\pi}\frac{-l\sin\theta\,\lambda}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|^3}l\,d\theta,\ \int_0^{2\pi}\frac{+z\lambda l}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|^3}d\theta\right)$$

$$= \left(0,\ 0,\ \frac{z\lambda l}{2\varepsilon_0(z^2+l^2)^{3/2}}\right)$$

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(2) $\vec{E}(0,0,z) = \left(0,\ 0,\ \int_0^R \frac{z l \sigma \, dl}{2\varepsilon_0 (z^2+l^2)^{\frac{3}{2}}}\right)$

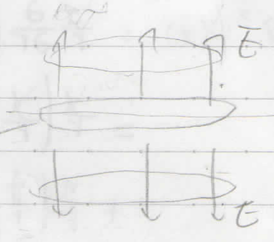
$$= \left(0,\ 0,\ \left[-\frac{z}{2\varepsilon_0}\frac{\sigma}{\sqrt{z^2+l^2}}\right]_0^R\right)$$

$$= \left(0,\ 0,\ \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+l^2}}\right)\right)$$

(3) $z/R \to 0$ のとき $(0,0,z)$ での電場は，$R\to\infty$ での面密度 σ の一様に電荷が分布した平面上の電場とみなすことができるので，

$$2\times\Delta S\times\vec{E}(0,0,z) = \frac{\sigma\,\Delta S}{\varepsilon_0}$$

$$\vec{E}(0,0,z) = \frac{\sigma}{2\varepsilon_0}$$



(2)から $z/R \to 0$ のとき

$$\vec{E}(0,0,z) = \left(0,\ 0,\ \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{R}\left(1-\frac{1}{2}\frac{z^2}{R^2}\right)\right)\right)$$

$$= \left(0,\ 0,\ \frac{\sigma}{2\varepsilon_0}\right)$$

(4) (2)より $\vec{E}(0,0,z) = \left(0,\ 0,\ \frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{1+(\frac{R}{z})^2}}\right)\right)$

で z成分をテイラー展開

$$\frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{1+(\frac{R}{z})^2}}\right) = \frac{\sigma}{2\varepsilon_0}\left(1-\left(1-\frac{1}{2}\left(\frac{R}{z}\right)^2+O\left(\left(\frac{R}{z}\right)^4\right)\right)\right)$$

$$= \frac{\sigma}{4\varepsilon_0}\left(\frac{R}{z}\right)^2$$

ラージオーダー

テイラー展開で，$\left(\frac{R}{z}\right)^2$ の係数は $\frac{\sigma}{4\varepsilon_0}$ である

多重極展開で $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}'-\vec{r})}{|\vec{r}'-\vec{r}|^3}\,dS$

$$\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = (\vec{r}-\vec{r}')\,|\vec{r}-\vec{r}'|^{-3}$$
$$= (\vec{r}-\vec{r}')\left(|\vec{r}|^2 - 2\vec{r}\cdot\vec{r}' + |\vec{r}'|^2\right)^{-\frac{3}{2}}$$
$$= \frac{(\vec{r}-\vec{r}')}{|\vec{r}|^3}\left(1 - \frac{2\vec{r}\cdot\vec{r}'}{|\vec{r}|^2} + \frac{|\vec{r}'|^2}{|\vec{r}|^2}\right)^{-\frac{3}{2}}$$
$$\fallingdotseq \frac{(\vec{r}-\vec{r}')}{|\vec{r}|^3}\left(1 + \frac{3\vec{r}\cdot\vec{r}'}{|\vec{r}|^2}\right) \fallingdotseq$$

いま $\vec{r} = (0, 0, z)$ $\vec{r}' = (l\cos\theta,\ l\sin\theta,\ 0)$ $(0 \leqq l \leqq R)$

としたとき $\vec{r}\cdot\vec{r}' = 0$ より

$$\vec{E}(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0}\left(\int_0^{2\pi}\!\!\int_0^R -\frac{l\cos\theta}{|\vec{r}|^3}\,dl\,d\theta,\ \int_0^{2\pi}\!\!\int_0^R -\frac{l\sin\theta}{|\vec{r}|^3}\,dl\,d\theta,\ \int_0^{2\pi}\!\!\int_0^R \frac{z}{|\vec{r}|^3}\,dl\,d\theta\right)$$
$$= \frac{\sigma}{4\pi\varepsilon_0}\left(0,\ 0,\ \frac{\pi R^2}{z^2}\right)$$
$$= \left(0,\ 0,\ \frac{\sigma R^2}{4\pi\varepsilon_0 z^2}\right)$$

∴ 多重極展開によって予想される係数は $\frac{\sigma}{4\pi\varepsilon_0}$

テイラー展開で求められた $\left(\frac{R}{z}\right)^2$ の係数と

多重極展開によって予想される係数は同じ。

問7 $\vec{r}' = (0, 0, z)$, $\vec{r} = (x, y, 0)$ とおく

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{d\vec{r}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \qquad \left(\begin{aligned} d\vec{r}'\times(\vec{r}-\vec{r}') \\ = (-y\,dz,\ x\,dz,\ 0)\end{aligned}\right)$$
$$= \frac{\mu_0 I}{4\pi}\left(\int_{-\infty}^{\infty}\frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}dz,\ \int_{-\infty}^{\infty}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}dz,\ 0\right)$$

$$\int_{-\infty}^{\infty}\frac{r}{(r^2+z^2)^{3/2}}dz$$
$$= \frac{1}{r}\int_{-\infty}^{\infty}\frac{1}{\left(1+\left(\frac{z}{r}\right)^2\right)^{3/2}}dz$$

$\frac{z}{r} = \tan\theta$ とおくと $dz = r\frac{1}{\cos^2\theta}d\theta$

z	$-\infty \longrightarrow \infty$
θ	$-\frac{\pi}{2} \longrightarrow \frac{\pi}{2}$

$$= \frac{r}{r'}\left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{r}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left(\frac{-2\sin\alpha}{r},\ \frac{2\cos\alpha}{r},\ 0\right)$$
$$= \left(\frac{\mu_0 I}{2\pi}\frac{-y}{x^2+y^2},\ \frac{\mu_0 I}{2\pi}\frac{x}{x^2+y^2},\ 0\right)$$

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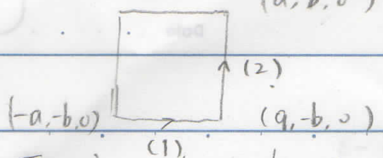
(2)



電流 $(-a, -b, 0) \longrightarrow (a, -b, 0)$ による磁界を $\vec{B}^{(1)}(\vec{r})$ とする

$\vec{B}^{(1)}(\vec{r}) \cdot \vec{I}^{(1)} = 0$ より

$\vec{B}^{(1)}(\vec{r}) = (0,\ B_y^{(1)}(\vec{r}),\ B_z^{(1)}(\vec{r}))$ とおける

$$\vec{B}_y^{(1)}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{-a}^{a}\frac{-z\,dx'}{((x-x')^2+(y+b)^2+z^2)^{\frac{3}{2}}}$$

$$\vec{B}_z^{(1)}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{-a}^{a}\frac{y\,dx'}{((x-x')^2+(y+b)^2+z^2)^{\frac{3}{2}}}$$

Biot-Savart の法則

電流 $(a, -b, 0) \longrightarrow (a, b, 0)$ による磁界を $\vec{B}^{(2)}(\vec{r})$ とする

$\vec{B}^{(1)}(\vec{r})$ と同様に $\vec{B}^{(2)}(\vec{r}) = (B_x^{(2)}(\vec{r}),\ 0,\ B_z^{(2)}(\vec{r}))$

$$\vec{B}_x^{(2)}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{-b}^{b}\frac{z\,dy'}{((x-a)^2+(y-y')^2+z^2)^{\frac{3}{2}}}$$

$$\vec{B}_z^{(2)}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{-b}^{b}\frac{-x\,dy'}{((x-a)^2+(y-y')^2+z^2)^{\frac{3}{2}}}$$

∴ 電流 $\vec{B}(\vec{r}) = 2(\vec{B}^{(1)}(\vec{r}) + \vec{B}^{(2)}(\vec{r}))$

$$= \left(\frac{\mu_0 I}{2\pi}\int_{-b}^{b}\frac{z\,dy'}{((x-a)^2+(y-y')^2+z^2)^{\frac{3}{2}}},\ \frac{\mu_0 I}{2\pi}\int_{-a}^{a}\frac{-z\,dx'}{((x-x')^2+(y+b)^2+z^2)^{\frac{3}{2}}},\ \frac{\mu_0 I}{2\pi}\int_{-a}^{a}\frac{y\,dx'}{((x-x')^2+(y+b)^2+z^2)^{\frac{3}{2}}} + \frac{\mu_0 I}{2\pi}\int_{-b}^{b}\frac{-x\,dy'}{((x-a)^2+(y-y')^2+z^2)^{\frac{3}{2}}}\right)$$

あとはわからん

自分でやってくれ!!

よし、じゃあ僕がやる!

⑨ 電磁気の授業にまともに出てないヤツがちょ〜しに乗って問題といてみた。
-ゲームやりてー

問題7 (2)

(1)から

$$\mathbb{B}(\mathbb{r}) = \left(-\frac{\mu_0 I}{2\pi}\frac{y}{x^2+y^2},\ \frac{\mu_0 I}{2\pi}\frac{x}{x^2+y^2},\ 0\right)$$

また

$$\mathbb{B}\cdot d\mathbb{r} = -\frac{\mu_0 I}{2\pi}\frac{y\,dx}{x^2+y^2} + \frac{\mu_0 I}{2\pi}\frac{x\,dy}{x^2+y^2}$$

(i) $(-a,-b,0) \longrightarrow (a,-b,0)$ の部分について。

$y=-b$, $dy=0$ に注意して線積分すると、

$$\int \mathbb{B}\cdot d\mathbb{r} = \int_{-a}^{a} -\frac{\mu_0 I}{2\pi}\frac{-b}{x^2+y^2}dx = \frac{\mu_0 I b}{2\pi}\int_{-a}^{a}\frac{dx}{x^2+y^2} = \frac{\mu_0 I}{2\pi b}\int_{-a}^{a}\frac{dx}{1+\left(\frac{x}{b}\right)^2} \cdots ①$$

$\frac{x}{b} = \tan\theta$ とすると $\left(dx = \frac{b}{\cos^2\theta}d\theta,\quad \begin{array}{c|ccc} x & -a & \longrightarrow & a \\ \hline \theta & \arctan\left(-\frac{a}{b}\right) & \longrightarrow & \arctan\left(\frac{a}{b}\right) \end{array} \right)$

$$① = \frac{\mu_0 I}{2\pi b}\int_{\arctan\left(-\frac{a}{b}\right)}^{\arctan\left(\frac{a}{b}\right)} \frac{1}{1+\tan^2\theta}\cdot\frac{b}{\cos^2\theta}d\theta = \frac{\mu_0 I}{2\pi}\int_{\arctan\left(-\frac{a}{b}\right)}^{\arctan\left(\frac{a}{b}\right)} d\theta = \underline{\frac{\mu_0 I}{2\pi}\left(\arctan\left(\frac{a}{b}\right) - \arctan\left(-\frac{a}{b}\right)\right)}_{②}$$

(ii) $(a,-b,0) \longrightarrow (a,b,0)$ の部分について。

$x=a$, $dx=0$ に注意して、同様にすると、

$$\int \mathbb{B}\cdot d\mathbb{r} = \underline{\frac{\mu_0 I}{2\pi}\left(\arctan\left(\frac{b}{a}\right) - \arctan\left(-\frac{b}{a}\right)\right)}_{③}$$

$(a,b,0) \longrightarrow (-a,b,0)$ は(i)と $(-a,b,0) \longrightarrow (-a,-b,0)$ は(ii)とそれぞれ結果が同じであるので、

$$\int_C \mathbb{B}\cdot\mathbb{r} = 2(②+③) = \frac{\mu_0 I}{\pi}\left(\underbrace{\arctan\left(\frac{a}{b}\right)+\arctan\left(\frac{b}{a}\right)}_{\frac{\pi}{2}} \underbrace{- \arctan\left(-\frac{b}{a}\right) - \arctan\left(-\frac{a}{b}\right)}_{\frac{\pi}{2}}\right)$$

$$= \frac{\mu_0 I}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \underline{\mu_0 I}$$

$\left(\because \arctan\left(\frac{a}{b}\right) + \arctan\left(\frac{b}{a}\right) = \frac{\pi}{2} \right.$

$\left. -\arctan(-\theta) = \arctan\theta \right)$